Solution to Assignment 3, MMAT5520

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Exercise 4.1:
3. **Soution:**

$$L[y] = y'' + 2t^{-1}y' + e^t y = 0.$$

$$W(y_1, y_2)(t) = ce^{-\int 2t^{-1}dt} = ct^{-2}.$$

Since $W(y_1, y_2)(1) = 3,\ c = 3$. Therefore, $W(y_1, y_2)(5) = \frac{3}{25}$.

Exercise 4.2:

1(b).**Soution:** We set $y = t^{-1}v$, then

$$\begin{aligned} y' &= t^{-1}v' - t^{-2}v, \\ y'' &= t^{-1}v'' - 2t^{-2}v' + 2t^{-3}v \end{aligned}$$

Thus the equation becomes

$$\begin{aligned} t^2(t^{-1}v'' - 2t^{-2}v' + 2t^{-3}v) + 4t(t^{-1}v' - t^{-2}v) + 2t^{-1}v &= 0, \\ tv'' + 2v' &= 0, \\ t^2v'' + 2tv' &= 0, \\ (t^2v')' &= 0, \\ t^2v' &= C, \\ v' &= Ct^{-2}, \\ v &= C_1t^{-1} + C_2. \end{aligned}$$

Therefore, $y = C_1t^{-2} + C_2t^{-1}$.

Exercise 4.3:
1(b).**Soution:** Solving the characteristic equation

$$r^2 + 9 = 0,$$

$$r = \pm 3i.$$

Thus the general solution is

$$y = C_1 \cos(3t) + C_2 \sin(3t).$$

1(d).**Soution:** The characteristic equation

$$r^2 - 8r + 16 = 0$$

has a double root $r_1 = r_2 = 4$. Thus the general solution is

$$y = C_1 e^{4t} + C_2 t e^{4t}.$$

1(e).**Soution:** Solving the characteristic equation

$$r^2 + 4r + 13 = 0,$$

$$r = -2 \pm 3i.$$

Thus the general solution is

$$y = e^{-2t}[C_1 \cos(3t) + C_2 \sin(3t)].$$

Exercise 4.4:

1(e).**Soution:** The characteristic equation $r^2 + 2r + 1 = 0$ has a double root -1. So the complementary function is

$$y_c = c_1 e^{-t} + c_2 t e^{-t}.$$

Since -1 is a double root of the characteristic equation, we let

$$y_p = t(At + B)e^{-t},$$

where A and B are constants to be determined. Now

$$y_p' = [-At^2 + (2A - B)t + B]e^{-t},$$
$$y_p'' = [At^2 - (4A - B)t + 2A - 2B]e^{-t}.$$

By comparing coefficients of

$$y_p'' + 2y_p' + y_p = 2e^{-t},$$
$$\{[At^2 - (4A - B)t + 2A - 2B] + 2[-At^2 + (2A - B)t + B] + At^2 + Bt\}e^{-t} = 2e^{-t},$$
$$A = 1.$$

We take $B = 0$, and a particular solution is

$$y_p = t^2 e^{-t}.$$

Therefore, the general solution is

$$y = y_c + y_p = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t}.$$

1(f).**Soution:** The characteristic equation $r^2 - 2r + 1 = 0$ has a double root 1. So the complementary function is

$$y_c = c_1 e^t + c_2 t e^t.$$

Since 1 is a double root of the characteristic equation, we let

$$y_p = A + t^2(Bt + C)e^t,$$

where A, B and C are constants to be determined. Now

$$
\begin{aligned}
y_p' &= [Bt^3 + (3B + C)t^2 + 2Ct]e^t, \\
y_p'' &= [Bt^3 + (6B + C)t^2 + (6B + 4C)t + 2C]e^t.
\end{aligned}
$$

By comparing coefficients of

$$
\begin{aligned}
y_p'' - 2y_p' + y_p &= te^t + 4, \\
\{[Bt^3 + (6B + C)t^2 + (6B + 4C)t + 2C] - 2[Bt^3 + (3B + C)t^2 + 2Ct] + Bt^3 + Ct^2\}e^{-t} + A &= te^t + 4, \\
A = 4,\ 6Bt + 2C &= t.
\end{aligned}
$$

We take $A = 4, B = \frac{1}{6}, C = 0$, and a particular solution is

$$y_p = \frac{1}{6}t^3e^t + 4.$$

Therefore, the general solution is

$$y = y_c + y_p = c_1e^t + c_2te^t + \frac{1}{6}t^3e^t + 4.$$

1(e).**Soution:** The characteristic equation $r^2 + 4 = 0$ has roots $\pm 2i$. So the complementary function is

$$y_c = c_1 \cos(2t) + c_2 \sin(2t).$$

Let

$$y_p = At^2 + Bt + C + De^t,$$

where A, B, C and D are constants to be determined. Now

$$
\begin{aligned}
y_p' &= 2At + B + De^t, \\
y_p'' &= 2A + De^t.
\end{aligned}
$$

By comparing coefficients of

$$
\begin{aligned}
y_p'' + 4y_p &= t^2 + 3e^t, \\
2A + De^t + 4(At^2 + Bt + C + De^t) &= t^2 + 3e^t, \\
4At^2 + 4Bt + 2A + 4C + 5De^t &= t^2 + 3e^t.
\end{aligned}
$$

We take $A = \frac{1}{4}, B = 0, C = -\frac{1}{8}, D = \frac{3}{5}$, and a particular solution is

$$y_p = \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t.$$

Therefore, the general solution is

$$y = y_c + y_p = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t.$$

2(a).**Soution:** The characteristic equation $r^2 + 3r = 0$ has roots $r = -3, 0$. So the complementary function is

$$y_c = c_1 + c_2e^{-3t}.$$

A particular solution takes the form

$$y_p = t(A_4t^4 + A_3t^3 + A_2t^2 + A_1t + A_0) + t(B_2t^2 + B_1t + B_0)e^{-3t} + C_1\sin(3t) + C_2\cos(3t).$$

2(b).**Soution:** The characteristic equation $r^2 - 5r + 6 = 0$ has roots $r = 2, 3$. So the complementary function is

$$y_c = c_1e^{2t} + c_2e^{3t}.$$

A particular solution takes the form

$$y_p = (A_1\cos(2t) + A_2\sin(2t))e^t + [(B_1t + B_0)\sin t + (C_1t + C_0)\cos t]e^{2t}.$$

2(c).**Soution:** The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$. So the complementary function is

$$y_c = c_1\sin t + c_2\cos t.$$

A particular solution takes the form

$$y_p = (A_1t + A_0) + t(B_1t + B_0)\sin t + t(C_1t + C_0)\cos t.$$

Exercise 4.5:
1(a).**Soution:** Solving the corresponding homogeneous equation, we let

$$y_1 = e^{2t},\ y_2 = e^{3t}.$$

We have

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t}.$$

So

$$\begin{aligned} u_1' &= -\frac{gy_2}{W} = -\frac{2e^te^{3t}}{e^{5t}} = -2e^{-t}, \\ u_2' &= \frac{gy_1}{W} = \frac{2e^te^{2t}}{e^{5t}} = 2e^{-2t}. \end{aligned}$$

Hence

$$\begin{aligned} u_1 &= 2e^{-t} + C_1, \\ u_2 &= -e^{-2t} + C_2, \end{aligned}$$

and the general solution is

$$y = u_1y_1 + u_2y_2 = e^t + C_1e^{2t} + C_2e^{3t}.$$

1(b).**Soution:** Solving the corresponding homogeneous equation, we let

$$y_1 = e^{2t},\ y_2 = e^{-t}.$$

We have

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} = -3e^t.$$

So

$$u_1' = -\frac{gy_2}{W} = -\frac{2e^{-t}e^{-t}}{-3e^t} = \frac{2}{3}e^{-3t},$$
$$u_2' = \frac{gy_1}{W} = \frac{2e^{-t}e^{2t}}{-3e^t} = -\frac{2}{3}.$$

Hence

$$u_1 = -\frac{2}{9}e^{-3t} + C_1,$$
$$u_2 = -\frac{2}{3}t + C_2,$$

and the general solution is

$$y = u_1y_1 + u_2y_2 = (C_2 - \frac{2}{9} - \frac{2}{3}t)e^{-t} + C_1e^{2t} = (C_3 - \frac{2}{3}t)e^{-t} + C_1e^{2t}.$$

Exercise 4.7:
1(c).**Soution:** The characteristic equation $r^4 - 2r^2 + 1 = 0$ has roots $r = \pm 1$(double roots). So the complementary function is

$$y_c = c_1e^t + c_2e^{-t} + c_3te^t + c_4te^{-t}.$$

A particular solution takes the form

$$y_p = t^2(A_1t + A_0)e^t.$$

1(e).**Soution:** The characteristic equation $r^4 + 2r^2 + 1 = 0$ has roots $r = \pm i$(double roots). So the complementary function is

$$y_c = c_1 \cos t + c_2 \sin t + c_3t \cos t + c_4t \sin t.$$

A particular solution takes the form

$$y_p = t^2(A_1t + A_0) \cos t + t^2(B_1t + B_0) \sin t.$$